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An analysis on the uncertainty of calculating the time constant of the quadrifilar reversed resistor

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Abstract

The uncertainty of calculating the time constant of the quadrifilar reversed resistor has been analyzed with the aim of determining the loss angle of a capacitor in terms of the calculated time constant. The results showed that the time constant of our resistor was calculated to be 1.31 $\times 10^{-9}$ s with a standard uncertainty of 6.0 $\times 10^{-10}$ s.

1. Introduction

Recently, precision impedance meters such as LCR meters have become commercially available and widely used in industry. One of the most important standards required for calibrating impedance meters is the capacitance standard. In response to industrial needs, the National Metrology Institute of Japan (NMIJ) has been engaged to establish a system for providing the capacitance standard [1].

Another important quantity for calibrating impedance meters is the dissipation factor or the loss angle of a capacitor. Industrial demand for a standard for the loss angle of a capacitor has been increasing year by year.

An absolute determination of the loss angle of a capacitor can basically be realized by a variable capacitor with adjustable electrode spacing [2)3)]. Although this method allows highly accurate measurements, it is not easy to realize the variable capacitor with sufficient accuracy. Another approach is to use a calculable AC resistor, with which the loss angle of a capacitor can be measured with reference to the time constant of a calculable AC resistor [4)-6)]. The NMIJ is planning to realize the standard for capacitor loss angle by using the latter method.

Before this method can be used, however, the calculation accuracy of the time constant must be investigated in order to know the measurement capability of the loss angle on the basis of the calculable time constant. This paper describes the analysis and estimates the uncertainty of determining the time constant of the quadrifilar reversed type of calculable resistor, as the basis for realizing the standard for capacitor loss angle.

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2. Time constant

Three types of calculable AC resistor are widely known: the coaxial type, the bifilar type and the quadrifilar reversed type. We used a quadrifilar reversed resistor of 10 kΩ for this analysis. Figure 1 shows a schematic drawing of the quadrifilar reversed resistor, which is constructed with a double loop of resistive wire and a cylindrical shield. This type of calculable resistor was originally developed by D. L. H. Gibbings [7)] and calculated in detail by J. Bohacek [4),8)]. According to their calculations, the time constant τ of the quadrifilar reversed resistor is given by

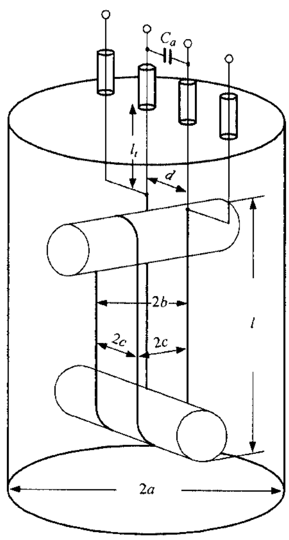

figure 1. Schematic drawing of the quadrifilar reversed resistor.

$$\tau = \frac{1}{R}\left(L - 8M_1 + 4M_2\right) - \frac{R}{6}\left(5C_1 + 3C_2 - C_0 + 6C_t + 6C_a\right) \quad (1)$$

where R is the resistance of the wire of 10 kΩ, L the self-inductance of the wire, M_1 the mutual inductance between adjacent wires, M_2 the mutual inductance between diagonally opposite wires, C_0 the capacitance between the total length of the wire and the outer shield, C_1 the total capacitance between adjacent wires, C_2 the total capacitance between diagonally opposite wires, C_t the capacitance between the terminal leads, and C_a the external capacitance added to the terminals for adjusting the time constant.

The self-inductance L and the mutual inductance M_1 and M_2 can be written as in [4]:

$$L \approx \frac{2\mu_0}{\pi} l \left(\ln\frac{2l}{r} - 1 + \frac{\mu_r}{4} \right) \quad (2)$$

$$M_1 \approx \frac{\mu_0}{2\pi} l \left[\ln\frac{l}{c} - 1 + \frac{2c}{l} - \frac{1}{4}\left(\frac{2c}{l}\right)^2 \right] \quad (3)$$

$$M_2 \approx \frac{\mu_0}{2\pi} l \left[\ln\frac{l}{b} - 1 + \frac{2b}{l} - \frac{1}{4}\left(\frac{2b}{l}\right)^2 \right] \quad (4)$$

where μ_0 is the permeability in vacuum of $4\pi \times 10^{-7}$, μ_r the relative permeability of the wire, and r the radius of the wire. Also, as shown in Figure 1, l is the folded length of the wire, $2b$ the wire spacing between diagonally opposite wires, and $2c$ the wire spacing between adjacent wires.

The capacitances C_0, C_1, C_2 in (1) are given by the following equations [4].

$$C_0 = \frac{4l}{p_1 + p_2 + 2p_3} \quad (5)$$

$$C_1 = \frac{p_3 l}{\left(p_1 + p_2\right)^2 - 4p_3^2} \quad (6)$$

$$C_2 = l\frac{p_2\left(p_1 + p_2\right) - 2p_3^2}{\left(p_1 - p_2\right)\left[\left(p_1 + p_2\right)^2 - 4p_3^2\right]} \quad (7)$$

where the potential coefficients are

$$p_1 = \frac{1}{2\pi\varepsilon_r\varepsilon_0}\ln\frac{a^2 - b^2}{ar} \quad (8)$$

$$p_2 = \frac{1}{2\pi\varepsilon_r\varepsilon_0}\ln\frac{a^2 + b^2}{2ab} \quad (9)$$

$$p_3 = \frac{1}{2\pi\varepsilon_r\varepsilon_0}\ln\frac{\sqrt{a^4 + b^4}}{2ac} \quad (10)$$

In (8)-(10), ε_0 is the permittivity in vacuum, ε_r the relative permittivity of the medium surrounding the wire, and a the radius of the cylindrical shield.

For the terminal capacitance of C_t, the same equation by which the capacitance between two wires of the bifilar type resistor can be calculated, is used. This was previously analyzed [6] and is given by

$$C_t = \frac{q_2 l_t}{q_1^2 - q_2^2} \quad (11)$$

where

$$q_1 = \frac{1}{2\pi\varepsilon_r\varepsilon_0}\ln\left(\frac{4a^2 - d^2}{4ar}\right) \quad (12)$$

$$q_2 = \frac{1}{4\pi\varepsilon_r\varepsilon_0}\ln\left[1 + \left(\frac{a}{d} - \frac{d}{4a}\right)^2\right] \quad (13)$$

As shown in Figure 1, d is the terminal lead spacing, and l_t is the length of the terminal leads.

The values listed in Table 1 were used to calculate the time

Table 1. Values of parameters used for calculating the time constant.

Parameter	Symbol	Value	Unit
Folded length of wire	l	407	mm
Wire radius	r	8	μm
Shield radius	a	47.5	mm
Wire spacing	b	14.1	mm
Wire spacing	c	10	mm
Length of terminal lead	l_t	5	mm
Terminal lead spacing	d	20	mm
Relative permeability of wire	μ_r	1	
Relative permittivity of medium	ε_r	1	
Potential coefficient	p_1	1.55×10^{11}	
Potential coefficient	p_2	1.08×10^{10}	
Potential coefficient	p_3	1.56×10^{10}	
Potential coefficient	q_1	1.55×10^{11}	
Potential coefficient	q_2	1.63×10^{10}	

constant. The inductances and capacitances required for (1) were obtained as shown in Table 2. Using (1), therefore, the time constant τ of the quadrifilar resistor in this analysis was calculated to be 1.31×10^{-9} s. To evaluate the uncertainty of this result, we analyzed the uncertainties of all values in Table 1.

Table 2. Values of resistance, inductance and capacitance of the quadrifilar reversed resistor.

Quantity	Symbol	Value	Unit
Resistance of wire	R	10.00	kΩ
Self-inductance of wire	L	3.51	μH
Mutual inductance between adjacent wires	M_1	0.22	μH
Mutual inductance between diagonally opposite wires	M_2	0.20	μH
Capacitance between wire and shield	C_0	8.28	pF
Capacitance between adjacent wires	C_1	0.24	pF
Capacitance between diagonally opposite wires	C_2	0.14	pF
Terminal capacitance	C_t	3.42	fF
External capacitance	C_a	1.00	pF
Time constant	τ	1.31×10^{-9}	s

3. Uncertainty analysis

With reference to the *Guide to the Expression of Uncertainty in Measurement*, the combined standard uncertainty of the time constant, $u_c(\tau)$, is obtained from (1), as

$$u_c^2(\tau)=\left[c_1u(R)\right]^2+\left[c_2u(L)\right]^2+\left[c_3u(M_1)\right]^2+\left[c_4u(M_2)\right]^2+\left[c_5u(C_0)\right]^2$$
$$+\left[c_6u(C_1)\right]^2+\left[c_7u(C_2)\right]^2+\left[c_8u(C_t)\right]^2+\left[c_9u(C_a)\right]^2 \qquad (14)$$

where $c_1, c_2, \cdots, c_9$ are the sensitivity coefficients which are given by

$$c_1=\frac{\partial\tau}{\partial R}$$
$$=-\frac{1}{R^2}(L-8M_1+4M_2)-\frac{1}{6}(5C_1+3C_2-C_0+6C_t+6C_a)$$
$$=8.06\times10^{-14} \qquad (15)$$

$$c_2=\frac{\partial\tau}{\partial L}=\frac{1}{R}=1.00\times10^{-4} \qquad (16)$$

$$c_3=\frac{\partial\tau}{\partial M_1}=-\frac{8}{R}=-8.00\times10^{-4} \qquad (17)$$

$$c_4=\frac{\partial\tau}{\partial M_2}=\frac{4}{R}=4.00\times10^{-4} \qquad (18)$$

$$c_5=\frac{\partial\tau}{\partial C_0}=\frac{R}{6}=1.67\times10^{3} \qquad (19)$$

$$c_6=\frac{\partial\tau}{\partial C_1}=-\frac{5R}{6}=-8.33\times10^{3} \qquad (20)$$

$$c_7=\frac{\partial\tau}{\partial C_2}=-\frac{R}{2}=-5.00\times10^{3} \qquad (21)$$

$$c_8=\frac{\partial\tau}{\partial C_t}=-R=-1.00\times10^{4} \qquad (22)$$

$$c_9=\frac{\partial\tau}{\partial C_a}=-R=-1.00\times10^{4} \qquad (23)$$

3.1. Resistance of the wire; $u(R)$

The resistance of the quadrifilar reversed resistor was calibrated against the Quantized Hall Resistance (QHR). The QHR system realized at the NMIJ provides calibrations of 10 kΩ resistance with a relative standard uncertainty of 0.03 μΩ/Ω. Thus, the standard uncertainty of the resistance of the wire, $u(R)$, was estimated to be 0.3 mΩ.

3.2. Self-inductance of the wire; $u(L)$

As shown in Table 1, the relative permeability of the wire was assumed to be μ_r = 1. Thus, from (2), the uncertainty in the derivation of the self-inductance of the wire, $u(L)$, can be written as

$$u^2(L)=\left[c_{21}u(l)\right]^2+\left[c_{22}u(r)\right]^2 \qquad (24)$$

where the sensitivity coefficients of c_{21} and c_{22} are

$$c_{21}=\frac{\partial L}{\partial l}=\frac{2\mu_0}{\pi}\left(\ln\frac{2l}{r}+\frac{\mu_r}{4}\right)=9.42\times10^{-6} \qquad (25)$$

$$c_{22} = \frac{\partial L}{\partial r} = -\frac{2\mu_0 l}{\pi r} = -4.07\times10^{-2} \tag{26}$$

The uncertainty components that contribute to $u(L)$ are restricted to the uncertainty of the folded length of the wire $u(l)$ and the uncertainty of the wire radius $u(r)$. The wires are folded on two supporting rods as shown in Figure 1, which can be moved for adjusting the strain of the wire. The distance between two supporting rods can easily be measured with an accuracy of 5 mm, so that the folded length of l was estimated to be within $\pm$ 5 mm. Also, the accuracy of the wire radius was estimated to be within $\pm$ 1 μm with reference to the manufacturer's specification. Assuming the rectangular distribution for each component, the standard uncertainties of the folded length $u(l)$ and the wire radius $u(r)$ were estimated to be 2.89 mm and 0.577 μm, respectively. Consequently, from (24), the standard uncertainty of $u(L)$ can be estimated to be 0.036 μH.

3.3. Mutual inductance between adjacent wires ; $u(M_1)$

From (3), the uncertainty in calculating the mutual inductance between adjacent wires, $u(M_1)$, can be written as

$$u^2(M_1) = \left[c_{31}u(l)\right]^2 + \left[c_{32}u(c)\right]^2 \tag{27}$$

where $c_{31}$and c_{32} are

$$c_{31} = \frac{\partial M_1}{\partial l} = \frac{\mu_0}{2\pi}\left[\ln\frac{l}{c} + \frac{1}{4}\left(\frac{2c}{l}\right)^2\right] = 7.41\times10^{-7} \tag{28}$$

$$c_{32} = \frac{\partial M_1}{\partial c} = -\frac{\mu_0}{2\pi}\left(\frac{l}{c} + \frac{2c}{l} - 2\right) = -7.75\times10^{-6} \tag{29}$$

The uncertainty components that contribute to $u(M_1)$ are the uncertainty of the folded length of wire $u(l)$ and the uncertainty of the wire spacing between adjacent wires $u(c)$. The estimation of $u(l)$ is the same as in 3.2. The probability distribution of the wire spacing $2c$ was assumed to be rectangular with an interval of 20 $\pm$5 mm. Thus, the standard uncertainty of $u(c)$ was estimated to be 1.44 mm, and the standard uncertainty of $u(M_1)$ was calculated from (27) to be 0.011 μH.

3.4. Mutual inductance between diagonally opposite wires; $u(M_2)$

From (4), the uncertainty of the mutual inductance between diagonally opposite wires, $u(M_2)$, is given by

$$u^2(M_2) = \left[c_{41}u(l)\right]^2 + \left[c_{42}u(b)\right]^2 \tag{30}$$

where c_{41} and c_{42} are

$$c_{41} = \frac{\partial M_2}{\partial l} = \frac{\mu_0}{2\pi}\left[\ln\frac{l}{b} + \frac{1}{4}\left(\frac{2b}{l}\right)^2\right] = 6.72\times10^{-7} \tag{31}$$

$$c_{42} = \frac{\partial M_2}{\partial b} = -\frac{\mu_0}{2\pi}\left(\frac{l}{b} + \frac{2b}{l} - 2\right) = -5.37\times10^{-6} \tag{32}$$

As in (30), two components, that is, the uncertainty of the folded length $u(l)$ which was estimated above, and the uncertainty of the wire spacing between diagonally opposite wires $u(b)$ contribute to $u(M_2)$. It is supposed that the wire spacing $2b$ has a rectangular distribution of 28$\pm$7.1 mm because the relation of $2b=\sqrt{2}\times2c$ exists as shown in Figure 1. Thus, the standard uncertainty of $u(b)$ was estimated to be 2.04 mm, and the standard uncertainty of $u(M_2)$ was calculated from (30) to be 0.011 μH.

3.5. Capacitance between the wire and the outer shield; $u(C_0)$

The uncertainty equation with respect to the derivation of C_0, which means the capacitance yielded between the total length of the wire and the shield, is given from (5), as

$$u^2(C_0) = \left[c_{51}u(p_1)\right]^2 + \left[c_{52}u(p_2)\right]^2 + \left[c_{53}u(p_3)\right]^2 + \left[c_{54}u(l)\right]^2 \tag{33}$$

where the sensitivity coefficients using the values listed in Table 1 are

$$c_{51} = \frac{\partial C_0}{\partial p_1} = -\frac{4l}{\left(p_1 + p_2 + 2p_3\right)^2} = -4.21\times10^{-23} \tag{34}$$

$$c_{52} = \frac{\partial C_0}{\partial p_2} = -\frac{4l}{\left(p_1 + p_2 + 2p_3\right)^2} = -4.21\times10^{-23} \tag{35}$$

$$c_{53} = \frac{\partial C_0}{\partial p_3} = -\frac{8l}{\left(p_1 + p_2 + 2p_3\right)^2} = -8.42\times10^{-23} \tag{36}$$

$$c_{54} = \frac{\partial C_0}{\partial l} = \frac{4}{p_1 + p_2 + 2p_3} = 2.03\times10^{-11} \tag{37}$$

Moreover, from (8)-(10), the following equations can be derived:

$$u^2(p_1) = \left[c_{101}u(a)\right]^2 + \left[c_{102}u(b)\right]^2 + \left[c_{103}u(r)\right]^2 \tag{38}$$

$$u^2(p_2) = \left[c_{201}u(a)\right]^2 + \left[c_{202}u(b)\right]^2 \tag{39}$$

$$u^2(p_3) = \left[c_{301}u(a)\right]^2 + \left[c_{302}u(b)\right]^2 + \left[c_{303}u(c)\right]^2 \tag{40}$$

where

$$c_{101} = \frac{\partial p_1}{\partial a} = \frac{a^2 + b^2}{2\pi\varepsilon_r\varepsilon_0 a\left(a^2 - b^2\right)} = 4.52\times10^{11} \tag{41}$$

$$c_{102} = \frac{\partial p_1}{\partial b} = -\frac{b}{\pi\varepsilon_r\varepsilon_0\left(a^2 - b^2\right)} = -2.47\times10^{11} \tag{42}$$

$$c_{103} = \frac{\partial p_1}{\partial r} = -\frac{1}{2\pi\varepsilon_r\varepsilon_0 r} = -2.25\times10^{15} \tag{43}$$

$$c_{201} = \frac{\partial p_2}{\partial a} = \frac{a^2 - b^2}{2\pi\varepsilon_r\varepsilon_0 a\left(a^2 + b^2\right)} = 3.17\times10^{11} \tag{44}$$

$$c_{202} = \frac{\partial p_2}{\partial b} = -\frac{a^2 - b^2}{2\pi\varepsilon_r\varepsilon_0 b\left(a^2 + b^2\right)} = -1.06\times10^{12} \tag{45}$$

$$c_{301} = \frac{\partial p_3}{\partial a} = \frac{a^4 - b^4}{2\pi\varepsilon_r\varepsilon_0 a\left(a^4 + b^4\right)} = 3.73\times10^{11} \tag{46}$$

$$c_{302} = \frac{\partial p_3}{\partial b} = \frac{b^3}{\pi\varepsilon_r\varepsilon_0\left(a^4 + b^4\right)} = 1.98\times10^{10} \tag{47}$$

$$c_{303} = \frac{\partial p_3}{\partial c} = -\frac{1}{2\pi\varepsilon_r\varepsilon_0 c} = -1.80\times10^{12} \tag{48}$$

Substituting (38)-(40) into (33), we obtain

$$\begin{aligned} u^2(C_0) = &\left[(c_{51}c_{101})^2 + (c_{52}c_{201})^2 + (c_{53}c_{301})^2\right]u^2(a) \\ &+\left[(c_{51}c_{102})^2 + (c_{52}c_{202})^2 + (c_{53}c_{302})^2\right]u^2(b) \\ &+\left[c_{51}c_{103}u(r)\right]^2 + \left[c_{53}c_{303}u(c)\right]^2 + \left[c_{54}u(l)\right]^2 \end{aligned} \tag{49}$$

For $u(C_0)$, therefore, five sources of uncertainties, which comprise the uncertainties of the shield radius $u(a)$, the wire spacing $u(b)$ and $u(c)$, the wire radius $u(r)$, and the folded length $u(l)$, must be analyzed. Four of five have already been estimated as described above. Supposing that the dimension of the cylindrical shield was manufactured with the accuracy of ±1 mm, then the standard uncertainty of $u(a)$ in the assumption of rectangular distribution was estimated to be 0.577 mm. Thus, the standard uncertainty of $u(C_0)$ can be derived from (49) to be 0.252 pF.

3.6. Capacitance between adjacent wires; $u(C_1)$

The uncertainty equation in the calculation of the total capacitance between adjacent wires, $u(C_1)$, can be derived from (6), as

$$u^2(C_1) = \left[c_{61}u(p_1)\right]^2 + \left[c_{62}u(p_2)\right]^2 + \left[c_{63}u(p_3)\right]^2 + \left[c_{64}u(l)\right]^2 \tag{50}$$

and the sensitivity coefficients are

$$c_{61} = \frac{\partial C_1}{\partial p_1} = -\frac{2p_3 l\left(p_1 + p_2\right)}{\left[\left(p_1 + p_2\right)^2 - 4p_3^2\right]^2} = -3.02\times10^{-24} \tag{51}$$

$$c_{62} = \frac{\partial C_1}{\partial p_2} = -\frac{2p_3 l\left(p_1 + p_2\right)}{\left[\left(p_1 + p_2\right)^2 - 4p_3^2\right]^2} = -3.02\times10^{-24} \tag{52}$$

$$c_{63} = \frac{\partial C_1}{\partial p_3} = l\frac{\left(p_1 + p_2\right)^2 + 4p_3^2}{\left[\left(p_1 + p_2\right)^2 - 4p_3^2\right]^2} = 1.66\times10^{-23} \tag{53}$$

$$c_{64} = \frac{\partial C_1}{\partial l} = \frac{p_3}{\left(p_1 + p_2\right)^2 - 4p_3^2} = 5.92\times10^{-13} \tag{54}$$

Substituting (38)-(40) into (50), we obtain

$$\begin{aligned} u^2(C_1) = &\left[(c_{61}c_{101})^2 + (c_{62}c_{201})^2 + (c_{63}c_{301})^2\right]u^2(a) \\ &+\left[(c_{61}c_{102})^2 + (c_{62}c_{202})^2 + (c_{63}c_{302})^2\right]u^2(b) \\ &+\left[c_{61}c_{103}u(r)\right]^2 + \left[c_{63}c_{303}u(c)\right]^2 + \left[c_{64}u(l)\right]^2 \end{aligned} \tag{55}$$

In (55), using the values estimated in the previous sections, the standard uncertainty of $u(C_1)$ was estimated to be 0.044 pF.

3.7. Capacitance between diagonally opposite wires; $u(C_2)$

From (7), the uncertainty of calculating the total capacitance between diagonally opposite wires, $u(C_2)$, is given by

$$u^2(C_2) = \left[c_{71}u(p_1)\right]^2 + \left[c_{72}u(p_2)\right]^2 + \left[c_{73}u(p_3)\right]^2 + \left[c_{74}u(l)\right]^2 \quad (56)$$

where

$$c_{71} = \frac{\partial C_2}{\partial p_1} = l\frac{p_2(p_1-p_2)\left[(p_1+p_2)^2-4p_3{}^2\right]-\left[p_2(p_1+p_2)-2p_3{}^2\right]\left[(p_1+p_2)^2-4p_3{}^2+2(p_1{}^2-p_2{}^2)\right]}{(p_1-p_2)^2\left[(p_1+p_2)^2-4p_3{}^2\right]^2}$$

$$= -1.57\times10^{-24} \quad (57)$$

$$c_{72} = \frac{\partial C_2}{\partial p_2} = l\frac{(p_1+2p_2)(p_1-p_2)\left[(p_1+p_2)^2-4p_3{}^2\right]+\left[p_2(p_1+p_2)-2p_3{}^2\right]\left[(p_1+p_2)^2-4p_3{}^2-2(p_1{}^2-p_2{}^2)\right]}{(p_1-p_2)^2\left[(p_1+p_2)^2-4p_3{}^2\right]^2}$$

$$= 1.81\times10^{-23} \quad (58)$$

$$c_{73} = \frac{\partial C_2}{\partial p_3} = -\frac{4p_3l(p_1+p_2)}{\left[(p_1+p_2)^2-4p_3{}^2\right]^2} = -6.05\times10^{-24} \quad (59)$$

$$c_{74} = \frac{\partial C_2}{\partial l} = \frac{p_2(p_1+p_2)-2p_3{}^2}{(p_1-p_2)\left[(p_1+p_2)^2-4p_3{}^2\right]} = 3.45\times10^{-13} \quad (60)$$

Substituting (38)-(40) into (56), we obtain

$$u^2(C_2) = \left[(c_{71}c_{101})^2+(c_{72}c_{201})^2+(c_{73}c_{301})^2\right]u^2(a) + \left[(c_{71}c_{102})^2+(c_{72}c_{202})^2+(c_{73}c_{302})^2\right]u^2(b) + \left[c_{71}c_{103}u(r)\right]^2+\left[c_{73}c_{303}u(c)\right]^2+\left[c_{74}u(l)\right]^2 \quad (61)$$

Therefore, the standard uncertainty of $u(C_2)$ was calculated to be 0.043 pF.

3.8. Capacitance between the terminal leads; $u(C_t)$

From (11), the uncertainty with respect to the terminal capacitance, $u(C_t)$ is given by

$$u^2(C_t) = \left[c_{81}u(q_1)\right]^2 + \left[c_{82}u(q_2)\right]^2 + \left[c_{83}u(l_t)\right]^2 \quad (62)$$

where the sensitivity coefficients are

$$c_{81} = \frac{\partial C_t}{\partial q_1} = -\frac{2q_1q_2l_t}{(q_1^2-q_2^2)^2} = -4.45\times10^{-26} \quad (63)$$

$$c_{82} = \frac{\partial C_t}{\partial q_2} = \frac{l_t(q_1^2+q_2^2)}{(q_1^2-q_2^2)^2} = 2.14\times10^{-25} \quad (64)$$

$$c_{83} = \frac{\partial C_t}{\partial l_t} = \frac{q_2}{q_1^2-q_2^2} = 6.84\times10^{-13} \quad (65)$$

Moreover, from (12) and (13), the following equations can be derived:

$$u^2(q_1) = \left[c_{401}u(a)\right]^2 + \left[c_{402}u(d)\right]^2 + \left[c_{403}u(r)\right]^2 \quad (66)$$

$$u^2(q_2) = \left[c_{501}u(a)\right]^2 + \left[c_{502}u(d)\right]^2 \quad (67)$$

where

$$c_{401} = \frac{\partial q_1}{\partial a} = \frac{4a^2+d^2}{2\pi\varepsilon_r\varepsilon_0 a(4a^2-d^2)} = 4.14\times10^{11} \quad (68)$$

$$c_{402} = \frac{\partial q_1}{\partial d} = -\frac{d}{\pi\varepsilon_r\varepsilon_0(4a^2-d^2)} = -8.34\times10^{10} \quad (69)$$

$$c_{403} = \frac{\partial q_1}{\partial r} = -\frac{1}{2\pi\varepsilon_r\varepsilon_0 r} = -2.25\times10^{15} \quad (70)$$

$$c_{501} = \frac{\partial q_2}{\partial a} = \frac{\left(\frac{a}{d}-\frac{d}{4a}\right)\left(\frac{1}{d}+\frac{d}{4a^2}\right)}{2\pi\varepsilon_r\varepsilon_0\left[1+\left(\frac{a}{d}-\frac{d}{4a}\right)^2\right]} = 3.46\times10^{11} \quad (71)$$

$$c_{502} = \frac{\partial q_2}{\partial d} = -\frac{\left(\frac{a}{d}-\frac{d}{4a}\right)\left(\frac{a}{d^2}+\frac{1}{4a}\right)}{2\pi\varepsilon_r\varepsilon_0\left[1+\left(\frac{a}{d}-\frac{d}{4a}\right)^2\right]} = -8.22\times10^{11} \quad (72)$$

Substituting (66) and (67) into (62), we obtain

$$u^2(C_t)=\left[(c_{81}c_{401})^2+(c_{82}c_{501})^2\right]u^2(a)$$
$$+\left[(c_{81}c_{402})^2+(c_{82}c_{502})^2\right]u^2(d)$$
$$+\left[c_{81}c_{403}u(r)\right]^2+\left[c_{83}u(l_t)\right]^2 \quad (73)$$

Therefore, to estimate the uncertainty of the terminal capacitance $u(C_t)$, the uncertainty due to the terminal lead spacing $u(d)$ and the uncertainty due to the length of the terminal leads $u(l_t)$ need to be evaluated. In the resistor used in this analysis, however, the terminal leads are mostly shielded in order to minimize the terminal capacitance of C_t. Thus, the uncertainty of $u(l_t)$ means the imperfection of the shielding for the terminal leads. Because the unshielded length of the terminal leads was estimated to be l_t = 5 ± 5 mm, the standard uncertainty of $u(l_t)$ was calculated to be 2.89 mm, assuming the rectangular distribution for l_t. Also, the spacing between the terminal leads was assumed to be d = 20 ± 10 mm in the rectangular distribution, so the standard uncertainty of $u(d)$ was estimated to be 5.77 mm. Consequently, the uncertainty of $u(C_t)$ was evaluated using (73) to be 2.22 fF.

3.9. External capacitance added to the terminals; $u(C_a)$

By externally connecting a variable capacitor to the terminals, the time constant of the resistor is reduced almost to zero. The capacitance added to the terminals was measured by using a precision capacitance meter (AH 2500), which was calibrated against the NMIJ standard of capacitance. From the measurements, the standard uncertainty of $u(C_a)$ was evaluated to be 0.58 fF.

Table 3 summarizes the results of the uncertainty analysis described above. Consequently, substituting these estimated values of uncertainties listed in Table 3 into (14), we determined the combined standard uncertainty $u_c(\tau)$ in the calculation of the time constant to be 6.0 ×10^{-10} s.

This result means that the phase angle of the resistor $\omega\tau$ can be determined with the standard uncertainty of 6.0×10^{-6} rad at an angular frequency of ω = 10^4 rad/s. Therefore, with the quadrature bridge that can relate resistance to capacitance with the relative standard uncertainty of a few parts in 10^8 [1], the loss angle of a capacitor is expected to be determined with the uncertainty of a few parts in 10^6 rad in terms of the calculable time constant.

Table 3. Uncertainty budget for determining the time constant of the quadrifilar reversed resistor.

Source of uncertainty	Symbol x_i	Type	Standard uncertainty $u(x_i)$	Sensitivity coefficient c_i	$\lvert c_i\rvert u(x_i)$
Resistance of wire	$u(R)$	A	0.3 mΩ	c_1 8.06×10^{-14}	0.00×10^{-10}
Self-inductance of wire	$u(L)$	B	0.036 μH	c_2 1.00×10^{-4}	0.04×10^{-10}
Mutual inductance between adjacent wires	$u(M_1)$	B	0.011 μH	c_3 -8.00×10^{-4}	0.09×10^{-10}
Mutual inductance between diagonally opposite wires	$u(M_2)$	B	0.011 μH	c_4 4.00×10^{-4}	0.04×10^{-10}
Capacitance between wire and shield	$u(C_0)$	B	0.252 pF	c_5 1.67×10^{3}	4.21×10^{-10}
Capacitance between adjacent wires	$u(C_1)$	B	0.044 pF	c_6 -8.33×10^{3}	3.67×10^{-10}
Capacitance between diagonally opposite wires	$u(C_2)$	B	0.043 pF	c_7 -5.00×10^{3}	2.15×10^{-10}
Terminal capacitance	$u(C_t)$	B	2.22 fF	c_8 -1.00×10^{4}	0.22×10^{-10}
External capacitance	$u(C_a)$	A	0.58 fF	c_9 -1.00×10^{4}	0.06×10^{-10}
Combined standard uncertainty	$u_c(\tau)$				6.0×10^{-10}

4. Conclusion

The time constant of the calculable AC resistor may be used as the standard of the capacitor loss angle. To derive a loss angle standard from the calculable time constant, we investigated the time constant of the quadrifilar reversed resistor and analyzed its calculation uncertainty. From the results, the standard uncertainty in calculating the time constant of the quadrifilar resistor can be estimated to be 6.0 ×10^{-10} s. Thus, the standard of the capacitor loss angle in terms of the calculable time constant will be achieved with the uncertainty of the order of 1 in 10^6 rad at ω = 10^4 rad/s.

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