Encoding method

Binary variables

$$H(q) = \sum_{i<j} Q_{ij} q_i q_j + \sum_i Q_{ii} q_i$$

Real variables

$$\Pi = \frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T \boldsymbol{F}$$

Encoding method: Representing real numbers with binary numbers

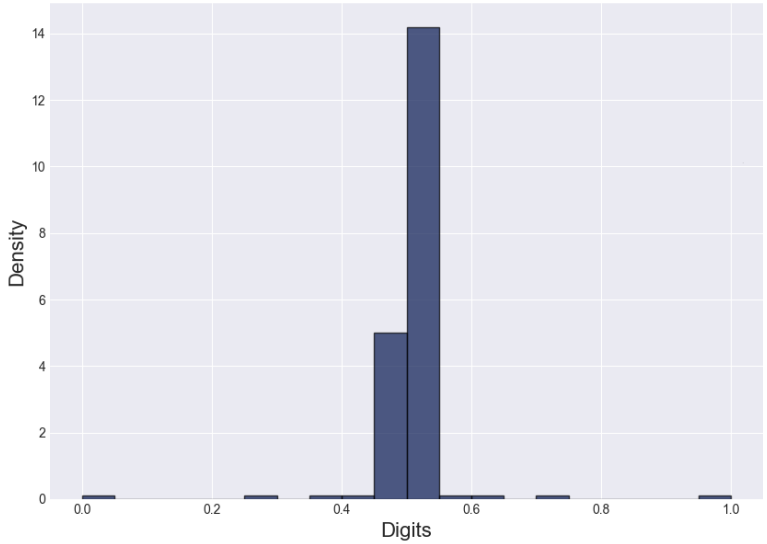
Encoding	Representation	Digits	Combinatorial effect
Binary	$x_i = \sum_{l=-m}^{m} 2^l q_{il}$	Exponential form with base 2	√
One-hot	$x_i = \sum_{l=0}^{m} r_l q_{il}$	Set uniformly	×
On-off[4]	$x_i = \sum_{l=0}^{m} r_l q_{il}$	Random sampling	√
Gaussian on-off[5]	$x_i = \sum_{l=0}^{m} r_l q_{il}$	Sampling with Gaussian distribution	√

[4] Endo, K., Matsuda, Y., Tanaka, S. *et al.* A phase-field model by an Ising machine and its application to the phase-separation structure of a diblock polymer. *Sci Rep* **12**, 10794 (2022).
[5] Endo, K., Matsuda, Y., Tanaka, S. *et al.* Novel real number representation in Ising machines and performance evaluation: Combinatorial random number sum and constant dicvision. Under review.

Images of digits distribution

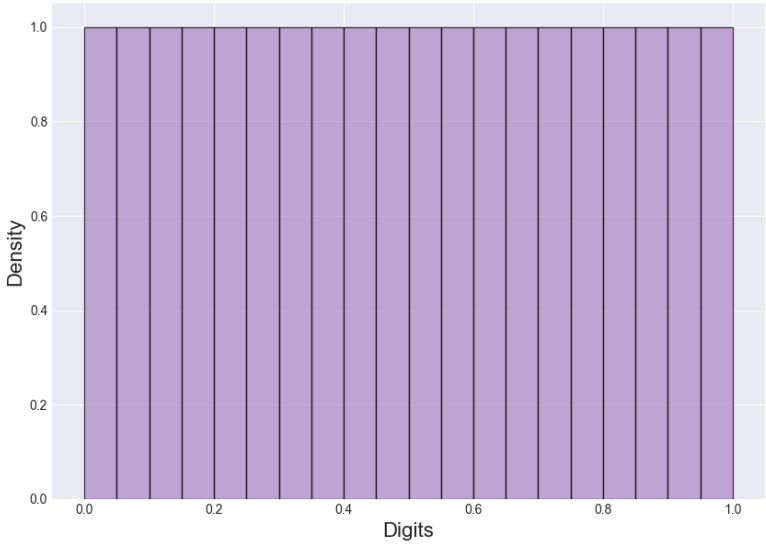
Binary

Exponential form with base 2



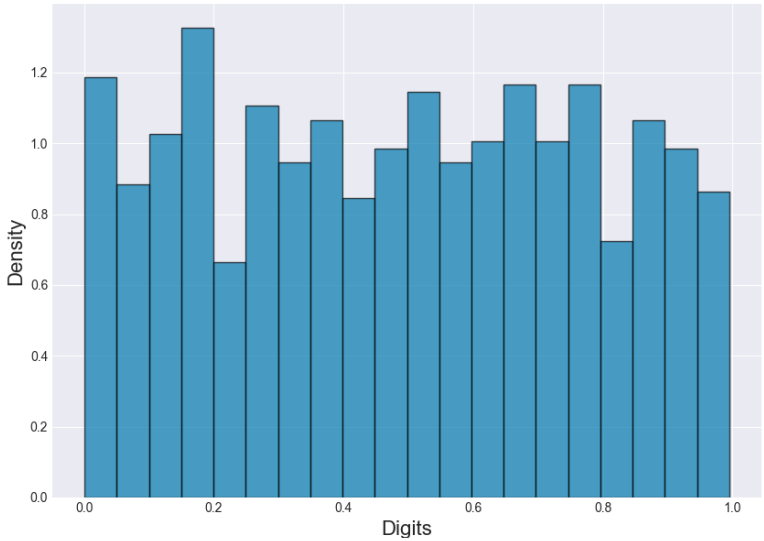
One-hot

Set uniformly



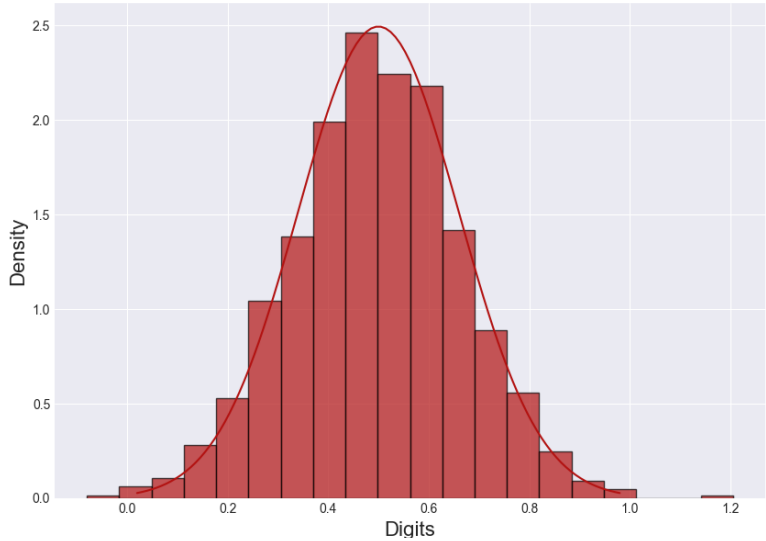
On-off

Random sampling



Gaussian on-off

Gaussian distribution



Gaussian on-off encoding

Gaussian on-off encoding represents real value x_i as

$$x_i = \sum_{l=1}^{m} r_l q_{i,l}, \tag{7}$$

where $\boldsymbol{r}(r_1, r_2, ..., r_m) \sim N(\mu, \sigma)$, which defines the selection of digits in Gaussian on-off encoding. The mean μ and standard deviation σ of the normal distribution are customized by the practitioner according to the real problem. In this study, the objective function of QUBO model is the total potential energy of truss analysis shown as

$$\Pi = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F} = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i x_j - \sum_{i=1}^{n} b_i x_i. \tag{8}$$

A is a symmetric matrix, hence the first term in Eq. 8 is

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i x_j = \frac{1}{2}\left(\sum_{i=1}^{n} a_{i,i}x_i^2 + 2\sum_{i<j} a_{i,j}x_i x_j\right). \tag{9}$$

By Gaussian on-off encoding, the first term of Eq. 9 can be expressed as

$$\frac{1}{2}\sum_{i=1}^{n} a_{i,i}x_i^2 = \frac{1}{2}\sum_{i=1}^{n} a_{i,i}\left(\sum_{l=1}^{m} r_l q_{i,l}\right)^2 = \frac{1}{2}\sum_{i=1}^{n} a_{i,i}\left(\sum_{l=1}^{m} r_l^2 q_{i,l} + 2\sum_{l_1<l_2} r_{l_1}r_{l_2}q_{i,l_1}q_{i,l_2}\right) \tag{10}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{m} a_{i,i}r_l^2 q_{i,l} + \sum_{i=1}^{n}\sum_{l_1<l_2} a_{i,i}r_{l_1}r_{l_2}q_{i,l_1}q_{i,l_2}, \tag{11}$$

and the second term of Eq. 9 denoted as

$$\sum_{i<j} a_{i,j}x_i x_j = \sum_{i<j} a_{i,j}\left(\sum_{l_1=1}^{m} r_{l_1}q_{i,l_1}\right)\left(\sum_{l_2=1}^{m} r_{l_2}q_{i,l_2}\right) = \sum_{i<j}\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} a_{i,j}r_{l_1}r_{l_2}q_{i,l_1}q_{i,l_2}. \tag{12}$$

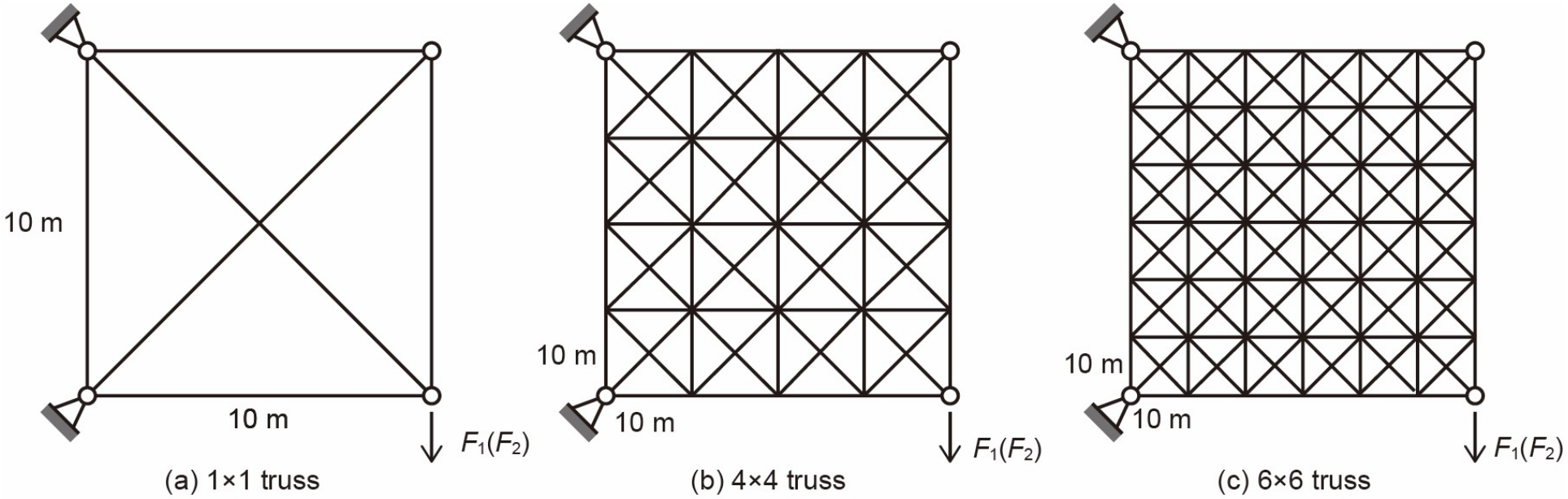
The second term of Eq. 8 is

$$\sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n}\sum_{l=1}^{m} b_i r_l q_{i,l}. \tag{13}$$

Joining Eq. 8, Eq. 11, Eq. 12 and Eq. 13, the form of Gaussian on-off encoding can be finally expressed as

$$\frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F} = \frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{m} a_{i,i}r_l^2 q_{i,l} + \sum_{i=1}^{n}\sum_{l_1<l_2} a_{i,i}r_{l_1}r_{l_2}q_{i,l_1}q_{i,l_2} + \sum_{i<j}\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} a_{i,j}r_{l_1}r_{l_2}q_{i,l_1}q_{i,l_2} - \sum_{i=1}^{n}\sum_{l=1}^{m} b_i r_l q_{i,l} \tag{14}$$

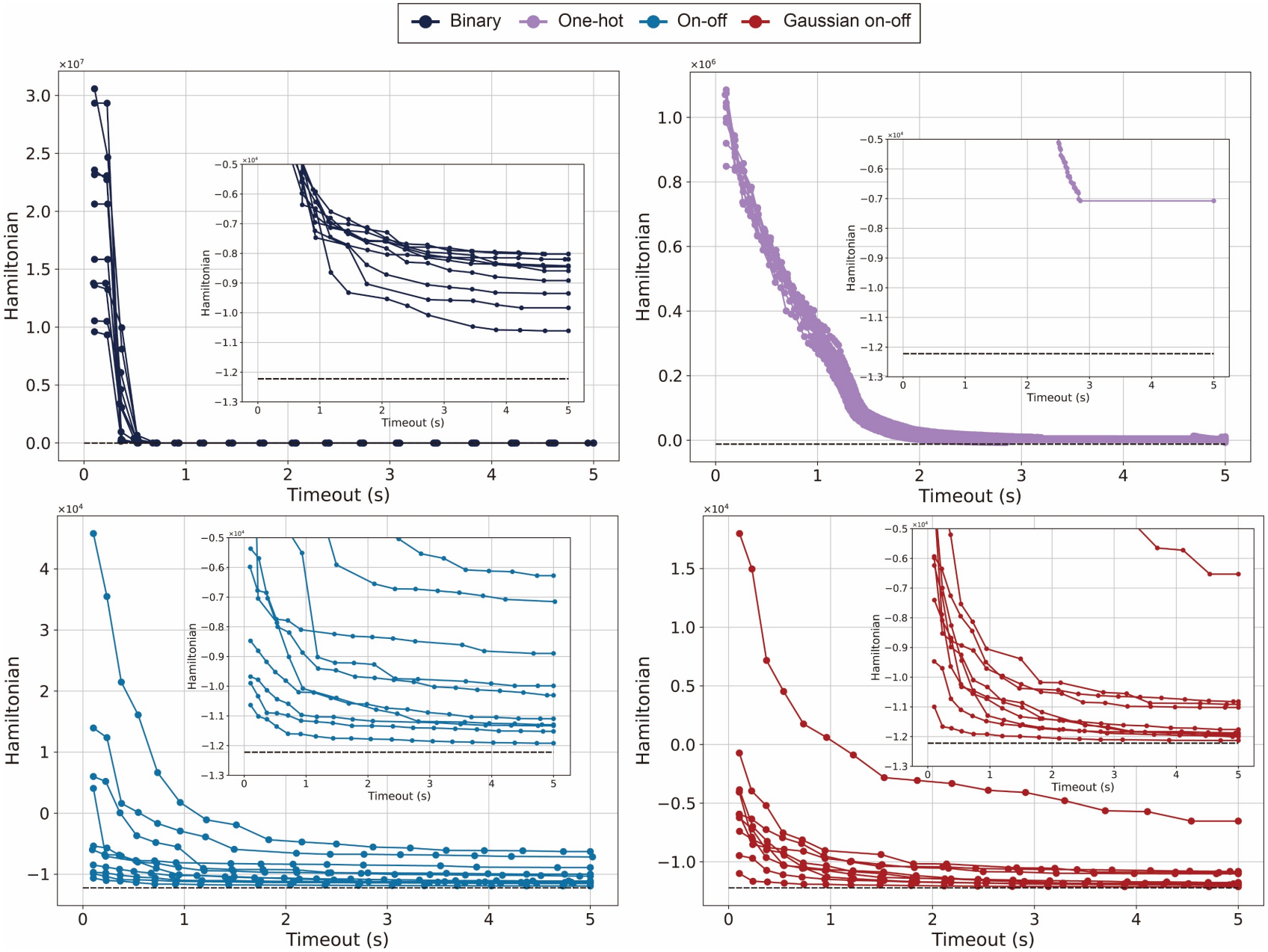
Truss structure



Initial conditions	Value
Length of each element	10 m
Young's modulus	2.0e+5 N/m^2
F_1	1.0e+4 N
F_2	5.0e+4 N

6×6 truss under F_1

Each scenario repeat for 10 times



Accuracy

Ratio of maximum displacement:

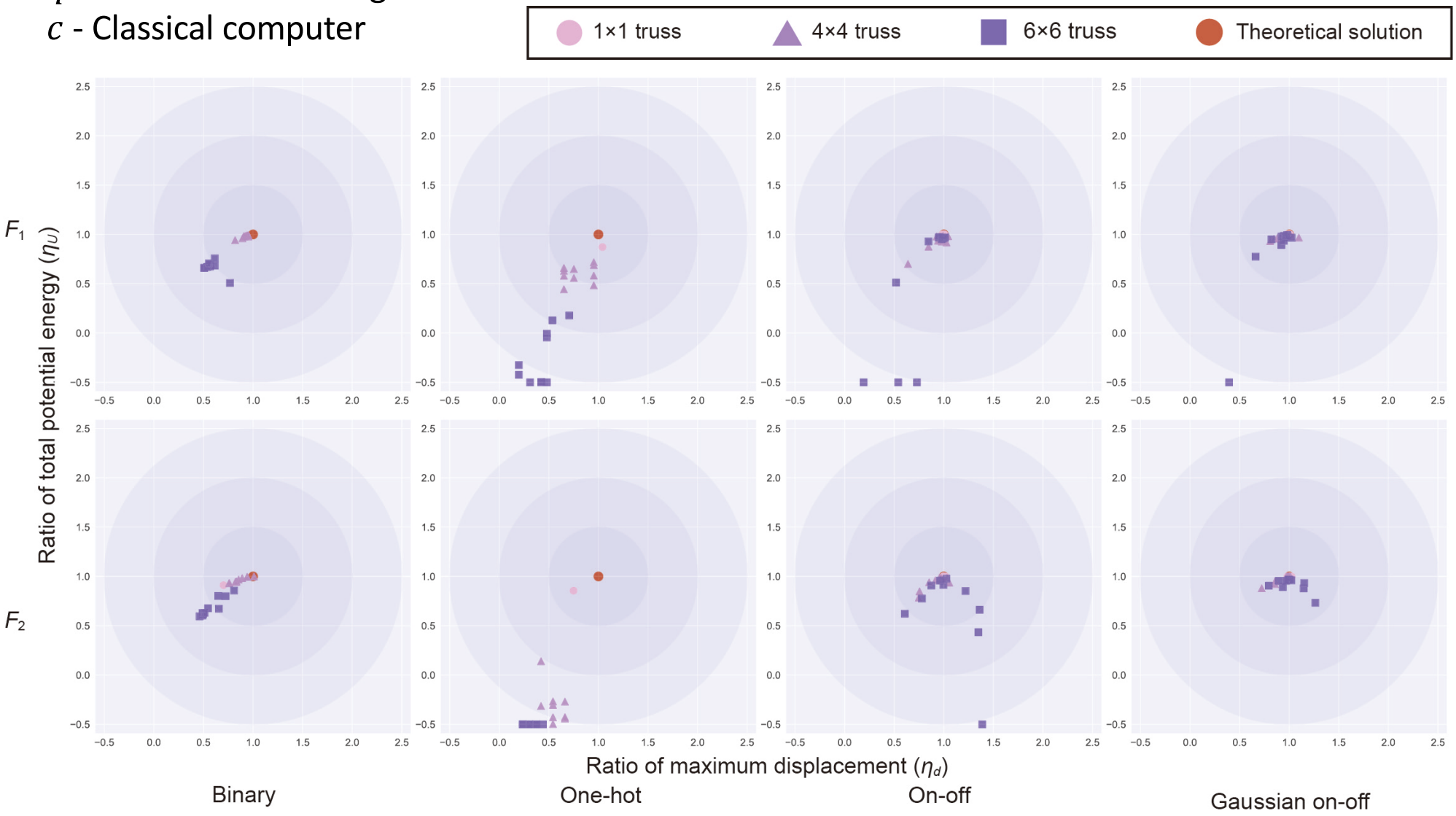
$$\eta_U = \frac{U_q}{U_c}$$

Ratio of total potential energy:

$$\eta_d = \frac{d_q}{d_c}$$

q - Quantum annealing

c - Classical computer



Best performance of each encoding method